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LETTER TO THE EDITOR

Confluent singularities in directed bond percolation

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Abstract. Existing series for directed bond percolation on the square, honeycomb and face centred cubic lattices are analysed with a view to detecting their confluent singularities. We compare our results with the confluent singularities in Reggeon field theory, which has been shown to be in the same universality class.

In recent months there has been a growing interest in the problem of directed bond percolation (DBP), which differs from the usual percolation in that passage across a given bond (allowed with probability p) is permitted only in one direction. DBP was first introduced by Broadbent and Hammersley (1957) and has potential applications to hopping transport in an electric field (Adler and Silver 1981), and galactic evolution (Schulman and Seiden 1980).

It has recently been shown (Cardy and Sugar 1980) that DBP is in the same universality class as Reggeon field theory (RFT), an effective field theory of high-energy physics (Migdal *et al* 1974a, b, Abarbanel and Bronzan 1974a, b, Abarbanel *et al* 1975, Moshe 1977). The classification of this theory in a class of models in chemistry and biology was also discussed by Grassberger and Sundermeyer (1978) and Grassberger and de la Torre (1979). The RFT of space dimension D (plus one 'time' dimension) is related to DBP on a $d = D+1$ dimensional acyclic directed lattice (see Blease (1977a) for DBP model details), and the direction of increasing 'time' in RFT corresponds to the special space or 'propagating' direction of the acyclic lattice.

The upper critical dimension for both models is $d_c = 5$ (Obukov 1980), and the exponents of the leading singularities in the percolation probability, $P(p)$, and mean cluster size, $S(p)$, of DBP (β and γ, respectively), as well as the exponents $\nu_\perp$ and $\nu_\parallel$ of the correlation length in the transverse and longitudinal directions (Domany and Kinzel 1981), are related to RFT exponents ν, z and η by the expressions (Cardy and Sugar 1980)

$$\beta = \tfrac{1}{2}\nu(\tfrac{1}{2}Dz - \eta), \tag{1}$$

$$\gamma = \nu(1+\eta), \tag{2}$$

$$\nu_\parallel = \nu, \qquad \nu_\perp = \nu_\parallel/\theta = \tfrac{1}{2}\nu z. \tag{3}$$

In RFT, the 'probability of percolation' from a point $(\mathbf{0}, 0)$ to a point $(\boldsymbol{x}, t)$ ($t = \ln s$, where s is the square of the centre of mass energy, and $\boldsymbol{x}$ is the D-dimensional impact parameter) is described by the two-point Green function

$$G(\boldsymbol{x}, t, \mathbf{0}, 0) = (p_c - p)^{\nu(Dz/2-\eta)}[\Phi_1(\Delta t, (p_c-p)^{\nu z}x^2) + (p_c-p)^{\nu\lambda}\Phi_2(\Delta t, (p_c-p)^{\nu z}x^2)].$$

Here Φ_1 is the scaling function of the dominant singularity, the mass term $\Delta \sim (p_c - p)^{\nu}$ is the inverse of the correlation length in the t direction, p_c plays the role of the critical percolation probability, and the scaling function Φ_2 and the exponent λ describe the confluent corrections. The scaling functions take different forms, depending on whether $p < p_c$ ($\alpha < \alpha_c$, $T > T_c$ in RFT terminology) or $p > p_c$. In the former case we can write the mean cluster size (the 'susceptibility' of a percolation model) as

$$S(p) \sim \chi(p) = \int \mathrm{d}t\, \mathrm{d}^D x\ G(\boldsymbol{x}, t)$$

and thus $S(p) \sim (p_c - p)^{-\gamma}[1 + a(p_c - p)^{\Delta_1}]$. For $p > p_c$ we have

$$G(\boldsymbol{x}, t) \sim P^2(p)\theta[v_0 t(p - p_c)^{\nu(1-z/2)} - |\boldsymbol{x}|]$$

where

$$P(p) \sim (p - p_c)^{\beta}[1 + b(p - p_c)^{\Delta_1}]$$

and $P(p)$ is the 'magnetisation' of a percolation model. From these identifications Cardy and Sugar (1980) found the relations of equations (1), (2) and (3), and we may further deduce that $\Delta_1 = \omega\nu$, the confluent exponent of the percolation process (see Aharony 1980), corresponds to $\lambda\nu$ in RFT ($\omega = \lambda$ is the derivative of the beta function at the fixed point).

To the best of our knowledge, no attempt has yet been made to calculate Δ_1 for DBP. In RFT, the exponents and scaling functions are related to measurable scattering cross sections (see the recent analysis in Baumel *et al* 1981); for example, the hadron–hadron total cross section $\sigma_T(s)$ behaves at high energies ($s \to \infty$) as

$$\sigma_T(s) = c_0(\ln s)^{\eta}[1 + c_1(\ln s)^{-\lambda} + \dots].$$

Since high-energy experiments are performed at large but finite energies s, one is interested in the non-leading terms in $\sigma_T(s)$ as well as in its asymptotic behaviour. Indeed, the approach to scaling as $s \to \infty$ which is governed by the exponent λ has been calculated in RFT at both $D = 1$ and $D = 2$ (Frazer and Moshe 1975a, b, Cardy 1977a, b, Brower *et al* 1978).

There is, in general, excellent agreement between DBP and RFT for the values of the critical exponents of the dominant singularities, and it would be of interest to ascertain whether this correspondence can be extended to the confluent corrections. The series of Blease (1977a, b, c) are suitable for this objective, and since analysis for confluent corrections requires initial series of substantial length, we have concentrated our efforts for $d = 2$ on his (1977c) pair connectedness series for the square and honeycomb lattices (since these are the longest available series). The moments $\mu_n(p)$ of the pair connectedness have the critical behaviour

$$\mu_n(p) \sim (p_c - p)^{\gamma - n\nu}[1 + a(p_c - p)^{\Delta_1}]$$

and for the square lattice we consider series for $n = 0, 2, 4$, whereas for the honeycomb only the series for $n = 0$ has been obtained. For $d = 3$ we study his $P(q)$ series ($q = 1 - p$) for the FCC lattice (Blease 1977a). The correction exponent should be the same in the series for $P(p)$, $S(p)$ and $\mu_n(p)$ in DBP and (as mentioned above) equal to $\lambda\nu$ of RFT.

We utilised three different Padé-type techniques, the following method giving the most stable results. We assume behaviour of the form

$$f_1(p) = A(p)(p_c - p)^g[1 + B(p_c - p)^{\Delta_1} + \ldots]$$
$$= A_1(p_c - p)^g[1 + B(p_c - p)^{\Delta_1} + A_2(p_c - p) + \ldots] \quad (4)$$

where the A_2 and higher-order terms arise from the Taylor expansion of $A(p)$, and construct the series for

$$f_2(p) = gf_1(p) + (p_c - p)\mathrm{d}f_1(p)/\mathrm{d}p \quad (5)$$

and

$$f_3(p) = (g+1)f_2(p) + (p_c - p)\,\mathrm{d}f_2(p)/\mathrm{d}p. \quad (6)$$

The logarithmic derivatives of f_1, f_2 and f_3 should have a pole at p_c with residues g, $g + \min\{\Delta_1, 1\}$ and $g + \min\{\Delta_1, 2\}$, respectively, and we searched for this pole both by forming Padé approximants to the logarithmic derivative of f_i, $(\lg f_i)'$, $i = 1, 2, 3$, and by evaluating the values of the Padé approximants to $(p - p_c)(\lg f_i)'$ at $p = p_c$ for $i = 2, 3$.

In order to calculate f_2 and f_3, we require input values of p_c and of g, the exponent of the dominant singularity. We re-analysed several series from Blease (1977a, c) and obtained good agreement with all his values, with the exception of the $d = 5$ hypercubic $S(p)$ series, where we found $p_c = 0.2085 \pm 0.0002$ and $\gamma = 1.119 \pm 0.005$. This value compares favourably with the RFT result of $\gamma = 1$ from the ε expansion (Baker 1974, Bronzan and Dash 1974a, b, c); the discrepancy is probably due to a combination of the logarithmic corrections at d_c and the shortness of the $S(p)$ series.

In the light of Blease's (1977a) discussion on the scaling relations, it is of interest to note that hyperscaling is built into the scaling functions of RFT. The hyperscaling relation $\gamma + 2\beta = d\nu$, which in DBP becomes (Kinzel and Yeomans 1981a, b) $\gamma + 2\beta = (D + \theta)\nu_\perp$, is an exact relation in RFT, as can be seen from equations (1)–(3). In DBP it is satisfied by the values obtained by Blease (1977a, b, c) within their error limits.

Our results for the two-dimensional lattices are presented in table 1 with RFT exponents included for purposes of comparison. The values for Δ_1 all agree (within their error limits) with the $\lambda\nu$ value of RFT (Brower *et al* 1978, Cardy 1977a, b). However, since Δ_1 is very close to 1.00, we are unable to determine conclusively whether we are observing the term with coefficient A_2 or B (equation (4)). This was also the case in the high-temperature series calculation in RFT (Brower *et al* 1978).

Table 1.

Lattice	Series	$\Delta_1 = \omega\nu$ (Our analysis)	P_c†	Leading exponent†
Square	$\mu_0(p)$	1.02 ± 0.02	0.6446 ± 0.0002	$\gamma = 2.271 \pm 0.016$
Square	$\mu_2(p)$	1.02 ± 0.02	0.6446 ± 0.0002	$\gamma + 2\nu = 5.730 \pm 0.030$
Square	$\mu_4(p)$	1.15 ± 0.15	0.6446 ± 0.0002	$\gamma + 4\nu = 9.195 \pm 0.044$
Honeycomb	$\mu_0(p)$	1.025 ± 0.025	0.8226 ± 0.0020	$\gamma = 2.250 \pm 0.021$
RFT‡ ($D = 1$)	$\chi(T)$	1.04 ± 0.02 ($\lambda\nu$)	$T_c = 0.60628 \pm 0.00004$	$\gamma = 2.286 \pm 0.020$
				$\nu = 1.736 \pm 0.001$

† p_c, γ, ν values for DBP are those of Blease (1977a, b, c) and are confirmed by our analysis.
‡ Values of Cardy (1977a, b) and Brower *et al* (1978).

However, in RFT there is an independent calculation by Cardy (1977a, b) that uses the high-order behaviour of perturbation theory, to obtain a value of $\lambda = 0.60 \pm 0.01$, and thus $\lambda\nu = 1.04 \pm 0.02$.

The Padé results were stable and 'perturbations' in the input values of p_c and γ within their error limits had no significant effect. Judging from our best results ($\mu_0(p)$ and $\mu_2(p)$ in table 1), we can state that $1.00 \leq \Delta_1 \leq 1.04$ in DBP ($d = 2$).

The case of the FCC lattice is unfortunately rather different. The error limits on p_c and β for the FCC lattice are relatively large; we do not have a reliable value of p_c for this lattice, since the $S(p)$ series are short, and we are unaware of any Monte Carlo calculations. In our analysis of f_2 (equation (5)) we found that the values of the confluent exponent are very sensitive to the input values of p_c and β, and from Blease's values of $\beta = 0.60 \pm 0.04$ and $p_c = 0.199 \pm 0.002$, we obtained a confluent exponent in the range $0.6 \leq \Delta_1 \leq 1.1$. For the high-temperature series of RFT in $D = 2$ ($d = 3$) the situation is even worse, since the series are too short to extract any value for $\lambda\nu$, but Cardy (1977a, b), employing, as mentioned above, the large-order behaviour of perturbation theory, obtains a value of $\lambda = 0.49 \pm 0.01$, and thus $\lambda\nu = 0.64 \pm 0.02$. Our estimate indeed includes this RFT value, but if we concentrate on the central values of p_c and β, we obtain confluent exponent results that are close to the upper limit of the above quoted range. Our analysis of f_3 (equation (6)) confirms this observation (although in this case the accuracy is low), and thus we cannot ignore the possibility that B is rather small and we have observed the A_2 term in our analysis. This ambiguity is reminiscent of the situation arising with the Ising models. Here, the susceptibility expansion takes the form

$$\begin{aligned}\chi(t) &\sim A(t)t^{-\gamma} + Bt^{-\gamma+\Delta_1} + \ldots \\ &= A_1 t^{-\gamma}(1 + A_2 t + Bt^{\Delta_1} + \ldots),\end{aligned}$$

and while initial results suggested that $B = 0$, $A_2 \neq 0$ for the $d = 3$ spin-$\frac{1}{2}$ model (Camp and Van Dyke 1975), it was later shown that $B \neq 0$ in accordance with the renormalisation group predictions (McKenzie 1979).

In summary, the analysis given above is consistent with universality between the confluent exponents of RFT and DBP.

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References

Abarbanel H D I and Bronzan J B 1974a *Phys. Lett.* **48B** 345
—— 1974b *Phys. Rev.* D **9** 2397
Abarbanel H D I, Bronzan J B, Sugar R L and White A R 1975 *Phys. Rep.* **21C** 119
Adler J and Silver M 1981 *Preprint*
Aharony A 1980 *Phys. Rev.* B **22** 400
Baker M 1974 *Phys. Lett.* **51B** 158
Baumel J, Feingold M and Moshe M 1981 *Technion-Haifa Preprint PH-81-28*
Blease J 1977a *J. Phys. C: Solid State Phys.* **10** 917
—— 1977b *J. Phys. C: Solid State Phys.* **10** 925
—— 1977c *J. Phys. C: Solid State Phys.* **10** 3461
Broadbent S R and Hammersley I M 1957 *Proc. Camb. Phil. Soc.* **53** 629

Bronzan J B and Dash J W 1974a *Phys. Lett.* **51**B 496
—— 1974b *Phys. Rev.* D **10** 4208
—— 1974c *Phys. Rev.* D **12** 1850
Brower R C, Furman M A and Moshe M 1978 *Phys. Lett.* **76B** 213
Camp W J and Van Dyke J P 1975 *Phys. Rev.* B **11** 2579
Cardy J L 1977a *Phys. Lett.* **67B** 97
—— 1977b *Private communication*
Cardy J L and Sugar R L 1980 *J. Phys. A: Math. Gen.* **13** L423
Domany E and Kinzel W 1981 *Weizmann Institute Preprint*
Frazer W R and Moshe M 1975a Phys. *Lett.* **57B** 475
—— 1975b *Phys. Rev.* D **12** 2370
Grassberger P and Sundermeyer K 1978 *Phys. Lett.* **77B** 220
Grassberger P and de la Torre A 1979 *Ann. Phys.* USA **122** 373
Kinzel W and Yeomans J 1981a *J. Phys. A: Math. Gen.* **14** L163
—— 1981b *Private communication*
McKenzie S 1979 *J. Phys. A: Math. Gen.* **12** L185
Migdal A A, Polyakov A M and Ter-Martirosyan K A 1974a *Phys. Lett.* B **48** 239
—— 1974b *Sov. Phys.-JETP* **40** 420
Moshe M 1977 *Phys. Rep.* **33C** 285
Obukov S P 1980 *Physica* **101A** 145
Schulman L S and Seiden P E 1980 *IBM Yorktown preprint*